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# A Fresh Look at Z-numbers – Relationships with Belief Functions and p-boxes

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## ABSTRACT

This paper proposes a new approach to the notion of Z-number, i.e., a pair of fuzzy sets modelling a probability-qualified fuzzy statement. Originally, a Z-number is viewed as the fuzzy set of probability functions stemming from the flexible restriction of the probability of a fuzzy event by a fuzzy probability. This representation leads to complex calculations and does not reduce to the original fuzzy event when the attached probability is 1. Simpler representations are proposed, that avoid these pitfalls. We note that when both fuzzy sets forming the Z-number are crisp, the generated set of probabilities is representable by a special kind of belief function that corresponds to a probability box (p-box). Two proposals are made to generalise this approach when the two sets are fuzzy. One approach considers a Z-number as a weighted family of crisp Z-numbers, obtained by independent cuts of the two fuzzy sets. In the alternative approach, a Z-number can be turned into a pair of possibility distributions forming a generalized p-box. In that case, the probability of each cut of the fuzzy event is upper and lower bounded by two probability values. Then computation with Z-numbers come down to uncertainty propagation with random intervals.

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## 1. Introduction

In order to account for uncertainty attached to fuzzy statements, Zadeh [1] introduced the notion of a Z-number (see also [2]). It was one of the last proposals made by him, that triggered a significant amount of literature. A Z-number is a pair  $(A, B)$ , where  $A$  is a fuzzy subset of  $U$  and  $B$  is a fuzzy subset of  $[0, 1]$  modelling a fuzzy restriction on the probability of  $A$ . Informally it tries to formalise the meaning of a statement of the form ‘ $(X \text{ is } A) \text{ has probability } B$ ’, like the statement ‘it is *probable* that my income will be *high* this year’. Zadeh introduced this kind of probability-qualified statements much earlier, when introducing a general framework for the mathematical representation of linguistic statements (the PRUF language [3]). The pair  $(A, B)$  was then modelling a probability-qualified statement, interpreted as a family of probability distributions obtained by fuzzily restricting the probability of the fuzzy event  $A$  by the fuzzy set  $B$ . The latter is viewed as a possibility distribution restricting the possible values of an ill-known probability. So a Z-number can be viewed as a fuzzy set of probability

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measures, more specifically a possibility distribution over them. In his paper, Zadeh [1] outlines a method to make computations with  $Z$ -numbers. This question has been taken up by other authors, most noticeably Aliev and colleagues [4–6], with a view to provide practical computation methods with  $Z$ -numbers. However, these methods seem to need the extensive use of linear programming even for solving small problems.

This research trend seems to have developed on its own with no connection with other generalised probabilistic frameworks for uncertainty management, like belief functions [7] and imprecise probabilities [8]. Yet, in those frameworks, the idea of a second-order possibility distribution over probability functions has been envisaged as a finer representation of convex probability sets by Moral [9] or Walley [10], with a view to represent linguistic information [11]. A full-fledged behavioural approach to the fuzzy probability of a crisp event (understood as partial knowledge about an objective probability) has been even studied in this spirit by De Cooman [12].

In this short paper, we reconsider  $Z$ -numbers and their interpretation by Zadeh in the light of belief functions and imprecise probabilities, questioning some choices made. First we characterise crisp  $Z$ -numbers where both  $A$  and  $B$  are crisp sets. This seems not to have been done yet. However, it is quite important to do it so as to put  $Z$ -numbers in the general perspective of uncertainty modelling. We show that a crisp  $Z$ -number can be exactly represented by a belief function [7], more specifically a  $p$ -box [13]. On such a basis, we first propose alternative interpretations of  $Z$ -numbers, which may sound more natural and make them easier to process in computations. Basically, we show that a  $Z$ -number can be represented, or at least approximated, by a belief function. As a consequence, computing with  $Z$ -numbers come down to computing with random intervals.

The paper is organised as follows. Section 2 recalls the original definition of a  $Z$ -number and highlights some limitations of this representation. Section 3 shows that in the crisp case, a  $Z$ -number is equivalently represented by a special kind of belief function. The next section presents the main results of the paper. First we try to extend the belief function view to the case when only one of the elements of the pair  $(A, B)$  is crisp. Finally, we deal with the general case, proposing two different approaches.

## 2. $Z$ -numbers

Consider two fuzzy sets  $A$  and  $B$ , where  $A$  is a fuzzy subset of the real line, typically a fuzzy interval, that stands for a fuzzy restriction on the value of some quantity  $X$ , and  $B$  is a fuzzy interval on  $[0, 1]$  that stands for a fuzzy probability. After Zadeh [1], a  $Z$ -number  $(A, B)$  represents the fuzzy set  $\tilde{\mathcal{P}}_{(A,B)}$  of probability measures  $P$  such that the probability  $P(A)$  of the fuzzy event  $A$  is fuzzily restricted by the fuzzy set  $B$ .

By definition, the scalar probability of fuzzy event  $A$  is  $P(A) = \int_0^1 \mu_A(x)p(x) dx$  [14], where  $p$  is the density of probability measure  $P$ . So the  $Z$ -number  $(A, B)$  represents a second-order possibility distribution over probability measures defined by

$$\forall P : \pi_{(A,B)}(P) = \mu_B(P(A)).$$

which is the membership function of  $\tilde{\mathcal{P}}_{(A,B)}$ . On this basis, one may compute the probability of some other fuzzy event  $C$  as the fuzzy probability  $\tilde{P}(C)$  such that

$$\mu_{\tilde{P}(C)}(p) = \sup_{P:P(C)=p} \pi_{(A,B)}(P) = \sup_{P:P(C)=p} \mu_B(P(C))$$

which defines an inferred Z-number  $(C, \tilde{P}(C))$ . Zadeh [1] calls  $Z^+$ -number, or bimodal distribution, the pair  $(A, P)$  where the probability distribution is made precise. It represents the statement ‘the probability that  $X$  is  $A$  is  $P(A)$ ’.

This interpretation of Z-number meets a difficulty: it seems we cannot recover as a special case the statement  $X$  is  $A$  expressed with full certainty, which should correspond to the Z-number  $(A, 1)$ , i.e.  $P(A) = 1$ . Indeed if  $A$  is a genuine fuzzy interval,  $P(A) = 1$  if and only if the support of  $P$  lies in the core of  $A$ ,  $\hat{A} = \{x : \mu_A(x) = 1\}$ . Indeed, it is clear that without this restriction, we always have  $P(A) < 1$ . To make sense of Zadeh’s interpretation of the Z-number, one should then accept the equivalence between  $(A, 1)$  and  $(\hat{A}, 1)$ , the latter being crisp. In other words, the original interpretation of Z-numbers loses the membership function of  $A$  on the way.

Zadeh [1] puts an additional restriction on the set of probability functions compatible with a Z-number, namely that the mean value of  $P$  be equal to the centroid of  $A$ . However, this restriction is questionable because the centroid of  $A$ , of the form  $\text{cent}(A) = \int_0^1 x \mu_A(x) / \int_0^1 \mu_A(x)$  comes down to considering the membership function as a probability distribution, the mean value of which is the centroid of  $A$ , thus doing away with the possibilistic understanding of  $A$  as representing incomplete information.

Zadeh [1] also deals with the computation of a Z-number  $(A_Z, B_Z)$  such that  $Z = f(X, Y)$  where the information on  $X$  is  $(A_X, B_X)$ , and on  $Y$  is  $(A_Y, B_Y)$ . Zadeh proposes to apply the extension principle to  $A_Z = f(A_X, A_Y)$ :

$$\mu_{f(A_X, A_Y)}(z) = \sup_{z=f(x,y)} \min(\mu_{A_X}(x), \mu_{A_Y}(y)),$$

and to define a possibility distribution  $\pi_{(A_Z, B_Z)}$  over the set of probability functions  $P_Z = P_X \circ_f P_Y$  obtained by probabilistic  $f$ -convolution of  $P_X$  and  $P_Y$  using the extension principle:

$$\pi_{(A_Z, B_Z)}(P_Z) = \sup_{P_X, P_Y: P_Z = P_X \circ_f P_Y} \min(\mu_{B_X}(P_X(A_X)), \mu_{B_Y}(P_Y(A_Y)))$$

where  $P_Z((-\infty, z]) = \int_{-\infty}^z (\int_{(x,y): u=f(x,y)} p_X(x) p_Y(y) dx dy) du$  (again requesting that  $E(P_X) = \text{Cent}(A_X)$ ,  $E(P_Y) = \text{Cent}(A_Y)$ ).

Then the recovering of a fuzzy probability  $B_Z$  is obtained by projection

$$\mu_{B_Z}(b) = \sup_{P_Z: P_Z(A)=b} \pi_{(A_Z, B_Z)}(P_Z)$$

This computation scheme is further studied by Yager [15] and Aliev et al. [4–6].

However, the proposed approach looks a bit questionable as (i) it is very complicated to implement and (ii) its formal justification is questionable. On the latter point, the fuzzy interval  $A$  represents the family of compatible probability functions  $\mathcal{P}(A) = \{P : P(C) \leq \Pi_A(C) = \sup_{u \in C} \mu_A(u), \forall C\}$  [16]. The above approach seems to consider that if probability measures  $P_1, P_2$  are, respectively, compatible with fuzzy intervals  $A_1$  and  $A_2$ , then  $P_Z = P_X \circ_f P_Y$  will be compatible with the fuzzy interval  $A_Z = f(A_X, A_Y)$ , which is quite unclear. For instance, it is well-known that the two conditions  $P_1 \in \mathcal{P}(A_X), P_2 \in \mathcal{P}(A_Y)$  do not imply that  $P_1 \circ_f P_2 \in \mathcal{P}(f(A_X, A_Y))$  [17, 18]. It is then dubious whether  $E(P_1) = \text{Cent}(A_X)$  and  $E(P_2) = \text{Cent}(A_Y)$  imply  $E(P_1 \circ_f P_2) = \text{Cent}(f(A_X, A_Y))$ .

The above discussion motivates the search for a different interpretation of Z-numbers, where the idea that it represents a set or a fuzzy set of probability measures is kept, but the use of a probability of fuzzy events is given up.

### 3. Crisp Z-numbers

Define a crisp Z-number to be a Z-number  $(A, B)$  where  $A$  is an interval on the real line<sup>1</sup> and  $B$  is a probability interval  $[b^-, b^+]$ . It is of interest to see what becomes of Zadeh's modelling of Z-numbers in this crisp case. It is clear that it yields the convex probability family  $\mathcal{P}(A, B) = \{P : b^- \leq P(A) \leq b^+\}$  and  $\pi_{(A,B)}$  is the characteristic function of this probability family. This set of probabilities can actually be described by a belief function on  $U$ . A belief function on a set  $U$  is defined by a mass assignment, that is an assignment  $m : \mathcal{F} \rightarrow (0, 1]$  of positive numbers to a finite family  $\mathcal{F}$  of subsets of  $U$ , whose elements  $E$  are called focal sets ( $m(E) > 0$ ), and such that  $\sum_{E \in \mathcal{F}} m(E) = 1$ .  $m$  is a probability distribution over  $2^U$ , namely over possible statements of the form  $X$  is  $E$ . The degree of belief  $Bel(C)$  in an event  $C$  induced by  $m$  is defined by  $Bel(C) = \sum_{E \in \mathcal{F}, E \subseteq C} m(E)$ . It is the probability that  $X$  is  $C$  can be inferred from some statement  $X$  is  $E$  for  $E \in \mathcal{F}$ . The degree of plausibility of an event  $C$  is  $Pl(C) = 1 - Bel(\bar{C}) \geq Bel(C)$ , where the overbar denotes set-complementation.

A belief function defined by  $m$  characterises a convex set of probabilities (called credal set)  $\mathcal{P}(m) = \{P : P(C) \geq Bel(C), \forall C \subseteq U\}$  in the sense that  $Bel(C) = \inf\{P(C) : P \in \mathcal{P}(m)\}$  is the lower probability induced by  $\mathcal{P}(m)$ . Given a crisp Z-number  $(A, [b^-, b^+])$ , consider the mass function defined by

$$m_{(A,B)}(A) = b^-; \quad m_{(A,B)}(\bar{A}) = 1 - b^+; \quad m_{(A,B)}(U) = b^+ - b^-.$$

In this model, the weight  $m_{(A,B)}(A)$  is the probability of knowing that  $X$  is  $A$ , the weight  $m_{(A,B)}(\bar{A})$  is the probability of knowing that  $X$  is  $\bar{A}$ , the weight  $m_{(A,B)}(U)$  is the probability of not knowing anything. The coincidence between this interpretation of the crisp Z-number with Zadeh's approach is summarised by the following easy-to-check proposition:

**Proposition 3.1:**  $\mathcal{P}(m_{(A,B)}) = \mathcal{P}(A, B)$ .

**Proof:** This because in this case,  $Bel(C) \in \{0, Bel(A), Bel(\bar{A})\}$  if  $C \neq U$ . Hence  $Bel(C) \leq P(C) \leq Pl(C)$  either trivially holds or comes down to  $Bel(A) = b^- \leq P(A) \leq Pl(A) = 1 - b^+$ . ■

A belief function  $Bel_1$  is said to be more committed (informative) than  $Bel_2$  if and only if  $Bel_1 \neq Bel_2$  and  $Bel_1 \geq Bel_2$ , which is clearly equivalent to  $\mathcal{P}(m_1) \subset \mathcal{P}(m_2)$ . The belief function induced by  $(A, B)$  via  $m_{(A,B)}$  is clearly the least committed belief function such that  $Bel(A) \geq b^-$  and  $Pl(A) \leq b^+$ . This is in conformity with Zadeh's intuition of Z-numbers. The Z-number is also akin to Shafer's view of belief functions as coming from an unreliable testimony [7]. Suppose a witness declares  $X$  is  $A$ , but you consider the statement as irrelevant with probability  $1 - b^-$ . It means that in the latter case the statement brings no information. This can be represented as a special case of a Z-number with  $b^+ = 1$ . The Z-number is then of the form  $(A, [b^-, 1])$ , which corresponds to a simple support belief function. The general case of a Z-number corresponds to an unreliable testimony where the witness may not only be irrelevant, but also has possibility of lying. Then, the probability that the witness is a liar is  $1 - b^+$ , while the probability that the information is irrelevant (brings no information) is  $b^+ - b^-$ . See Pichon et al. [19] for a study in the combination of such kind of testimonies, that can be useful for the merging of Z-numbers coming from several sources.

<sup>1</sup> More generally,  $A$  can be any subset of a referential set.

The crisp Z-number can be actually represented by a special family of belief functions that coincide with p-boxes [13]. A p-box is the set of probability measures on a totally ordered set whose cumulative function is limited by an upper and a lower cumulative function: there are distribution functions  $F^* \geq F_*$  that define the probability family  $\{P : F^*(u) \geq P(\inf U, u) \geq F_*(u)\}$ . The focal sets of the corresponding belief function are in the form  $E_\alpha = [\inf\{u : F^*(u) \geq \alpha\}, \inf\{u : F_*(u) \geq \alpha\}]$ , (that is, if the cumulative functions are continuous bijections,  $[(F^*)^{-1}(\alpha), (F_*)^{-1}(\alpha)]$ ,  $0 < \alpha \leq 1$ ). To see it, let  $A = [a^-, a^+] \subset U = [a^-, u^+]$ . Then we have:

$$F_*(x) = Bel([u^-, x]) = \begin{cases} 0 & \text{if } x < a^+ \\ b^- & \text{if } a^+ \leq x < u^+ \\ 1 & \text{if } x = u^+ \end{cases},$$

$$F^*(x) = Pl([u^-, x]) = \begin{cases} b^+ & \text{if } a^- \leq x < a^+ \\ 1 & \text{if } a^+ \leq x \leq u^+ \end{cases}.$$

We recover the three focal sets of  $m_{(A,B)}$  as  $A$  for  $0 \leq \alpha < b^-$ ,  $U$  for  $b^- \leq \alpha < b^+$ , and  $\bar{A}$  for  $b^+ \leq \alpha < 1$ . Note that for finite  $U$ , it is always possible to rank-order the  $n$  elements of  $U$  such that  $A = \{u_1, \dots, u_k\}$ , and  $\bar{A} = \{u_{k+1}, \dots, u_n\}$  so that the focal sets are of the form  $A = [u_1, u_k]$ ,  $U = [u_1, u_n]$ ,  $\bar{A} = [u_{k+1}, u_n]$ .

However, requesting the additional condition  $E(P) = Cent(A)$  comes down to restricting to probability distributions in  $\mathcal{P}(m_{(A,B)})$  such that the mean value  $E(P)$  is the midpoint of interval  $[a^-, a^+]$ . This linear constraint, notwithstanding its lack of natural justification (it does not even imply the symmetry of the density of  $P$ ), leads to a smaller credal set that cannot generally be represented by a belief function, which significantly increases the complexity of handling Z-numbers in practice.

Proposition 3.1 makes the computation with crisp p-boxes quite easy to perform. Let  $m_X$  and  $m_Y$  be the mass functions induced by two independent crisp Z-numbers  $(A_X, B_X)$  and  $(A_Y, B_Y)$ , respectively. We can apply the random set propagation method (Yager [20], Dubois and Prade [21, 22]) to compute the mass function for  $f(X, Y)$  as

$$m_{f(X,Y)}(G) = \sum_{E_X \in \mathcal{F}_X, E_Y \in \mathcal{F}_Y : G = f(E_X, E_Y)} m_X(E_X) \cdot m_Y(E_Y).$$

This is not computationally extensive as each belief function only has three focal sets. However, it is easy to figure out that in general the result will not always be represented by a Z-number as the result potentially has nine focal sets.

For instance, consider the sum on  $\mathbb{R}$  of two crisp Z-numbers  $([a_1^-, a_1^+], [b_1^-, b_1^+])$ , and  $([a_2^-, a_2^+], [b_2^-, b_2^+])$ . Suppose without loss of generality that  $a_1^- + a_2^+ < a_1^+ + a_2^-$ . Note that  $\bar{A}_i = (-\infty, a_i^-) \cup (a_i^+, +\infty)$ . Computing  $E_X + E_Y$  yields focal sets

- $A_1 + A_2 = [a_1^- + a_2^-, a_1^+ + a_2^+]$  with mass  $b_1^- b_2^-$ .
- $\bar{A}_1 + A_2 = (-\infty, a_1^- + a_2^+) \cup (a_1^+ + a_2^-, +\infty)$ , with mass  $(1 - b_1^+) b_2^-$
- $\mathbb{R}$  otherwise, with mass  $1 - b_2^- (1 + b_1^- - b_1^+)$

It does not correspond to a Z-number because  $\bar{A}_1 + A_2 \neq \overline{A_1 + A_2}$  in general. However, it is possible to extract a Z-number  $(A', B')$  from the result, where the idea is to compute the



probability interval  $B'$  given  $A'$ . For instance, in the above example, we get the Z-number  $(A_1 + A_2, [b_1^- b_2^-, 1])$  since  $Bel(A_1 + A_2) = b_1^- b_2^-$ , and  $Pl(A_1 + A_2) = 1$  since the two first focal sets overlap. It is clear that the latter Z-number is only part of the whole information obtained by computing the sum  $X + Y$ .

#### 4. Interpreting Z-numbers : Various Proposals

In the following, we see to what extent general Z-numbers can be interpreted by belief functions, and whether this point of view on Z-numbers is in agreement or not with Zadeh's approach relying on probabilities of a fuzzy event. We first deal with cases where one of the components of  $(A, B)$  is crisp, the other being fuzzy.

##### 4.1. Crisp Probability Qualification of Fuzzy Events: $(A, [b^-, b^+])$

Consider the case of a fuzzy statement  $X$  is  $A$  whose probability is considered to lie in  $[b^-, b^+]$ . After Zadeh [1], it corresponds to a probability set  $\mathcal{P}_{(A,B)} = \{P : b^- \leq P(A) \leq b^+\}$ , where  $P(A)$  is the probability of a fuzzy event  $A$ . It is a convex probability set characterised by linear constraints. For instance,  $A$  could be a fuzzy subset of a finite set  $U = \{u_1, \dots, u_n\}$  and the constraint is of the form  $b^- \leq \sum_{i=1}^n \mu_A(u_i) p_i \leq b^+$  on probability assignments  $(p_1, \dots, p_n)$ . It does not characterise a belief function at all [8].

*Z-number and fuzzy belief structures.* However, there is another interpretation of the Z-number, whereby the weight  $b^-$  is assigned not to the fuzzy event  $A$ , but to the fact of *knowing* that  $X$  is  $A$  and nothing more. Then what we get is a fuzzy belief structure, first proposed by Yen [23], which is a belief function whose focal sets are fuzzy and form a family  $\tilde{\mathcal{F}}$ . Formally, we still have a mass function  $\tilde{m}$  such that  $\tilde{m}(A) = b^-$ ,  $\tilde{m}(\bar{A}) = 1 - b^+$  and  $\tilde{m}(U) = b^+ - b^-$  where the fuzzy focal set  $\bar{A}$  has membership function  $\mu_{\bar{A}} = 1 - \mu_A$ . Then the definitions of belief and plausibility functions are extended as follows:

$$Bel(C) = \sum_{F \in \tilde{\mathcal{F}}} \left( \min_{x \notin C} 1 - \mu_F(x) \right) \tilde{m}(F); \quad Pl(C) = \sum_{F \in \tilde{\mathcal{F}}} \left( \max_{x \in C} \mu_F(x) \right) \tilde{m}(F).$$

Here,

$$Bel(C) = \left( \min_{x \notin C} 1 - \mu_A(x) \right) \tilde{m}(A) + \left( \min_{x \notin C} \mu_A(x) \right) \tilde{m}(\bar{A}) \quad (1)$$

$$= \left( \min_{x \notin C} 1 - \mu_A(x) \right) b^- + \left( \min_{x \notin C} \mu_A(x) \right) (1 - b^+); \quad (2)$$

$$Pl(C) = \left( \max_{x \in C} \mu_A(x) \right) b^- + \left( \max_{x \in C} 1 - \mu_A(x) \right) (1 - b^+) + b^+ - b^- \quad (3)$$

If the fuzzy focal sets have a finite number of membership grades  $\alpha_1 = 1 > \alpha_2 > \dots > \alpha_k > 0$ , and the  $\alpha_i$ -cut of the focal set  $F$  is  $F_{\alpha_i} = \{u : \mu_F(u) \geq \alpha_i\}$ , it can be checked that the belief and plausibility functions defined above derive from the mass assignment whose focal sets are  $\alpha$ -cuts of fuzzy focal sets: they form the family  $\mathcal{F} = \bigcup_{F \in \tilde{\mathcal{F}}} \{F_{\alpha_i} : i = 1, \dots, k\}$  with mass  $m(F_{\alpha_i}) = \tilde{m}(F)(\alpha_i - \alpha_{i+1})$  for all fuzzy focal

sets  $F$ . Then  $\sum_{i=1,k} m(F_{\alpha_i}) = \tilde{m}(F)$  and it is easy to verify the following claim.

**Proposition 4.1:**  $Bel(C) = \sum_{F \in \tilde{\mathcal{F}}} (\min_{x \notin C} 1 - \mu_F(x)) \tilde{m}(F) = \sum_{i=1}^k \sum_{F \in \tilde{\mathcal{F}}, F_{\alpha_i} \subseteq C} m(F_{\alpha_i})$ .

Under this view, the Z-number  $(A, [b^-, b^+])$  can be expressed by the belief function with mass assignment  $m$  that allocates weight  $m(F_{\alpha_i}) = b^-(\alpha_i - \alpha_{i+1})$  to cuts  $F_{\alpha_i}$  of  $F$  and  $m(\bar{F}_{\alpha_i}) = (1 - b^+)(\alpha_i - \alpha_{i+1})$  to the cuts  $\bar{F}_{\alpha_i}$  of their complements, while  $m(U) = \tilde{m}(U)$ .

The natural question is whether  $\mathcal{P}(m) = \{P : P(C) \geq Bel(C)\} = \mathcal{P}_{(A, [b^-, b^+])}$  namely if the credal set induced according to Zadeh's view is in agreement with the belief function approach. In other words, does Proposition 3.1 still hold when  $A$  is fuzzy? The answer is no since as pointed out above,  $\mathcal{P}(A, [b^-, b^+])$  is not induced by constraints on the probabilities of events only, but by linear constraints [8].

Besides, consider again the special case where the interval  $[b^-, b^+]$  reduces to the value 1. As seen earlier,  $\mathcal{P}(A, 1)$  is equivalent to the set  $\mathcal{P}(\hat{A}, 1)$ , where  $\hat{A}$  is the core of  $A$ . Thus, Zadeh's approach to Z-numbers interprets  $(A, 1)$  as the statement  $X$  is  $\hat{A}$ , which sounds questionable. In contrast, the present approach interprets  $(A, 1)$  as a belief function whose focal sets are the cuts of  $A$  (a necessity measure based on a possibility distribution  $\pi = \mu_A$  in the spirit of Zadeh [24]). Using the fuzzy belief structure view, one has that  $(A, 1)$  naturally reduces to the statement  $X$  is  $A$ , and the probability family is  $\mathcal{P}(A)$ , the one induced by  $\pi = \mu_A$  previously defined in Section 2.

*Z-number as a parametric belief function.* An alternative interpretation of the Z-number  $(A, [b^-, b^+])$  can be a *fuzzy set of standard belief functions*: we consider the Z-number  $(A, [b^-, b^+])$  as a parameterised set of crisp Z-numbers  $Z_\alpha = (A_\alpha, [b^-, b^+])$ ,  $\alpha \in (0, 1]$ . Each  $Z_\alpha$  gives a mass function  $m_\alpha$  with focal sets  $A_\alpha, \bar{A}_\alpha, U$  with respective weights  $b^-, 1 - b^+, b^+ - b^-$ . Note that each value  $\alpha$  corresponds to a single belief function  $Bel_\alpha$ , i.e. this construction rather yields a gradual element (in the sense of [25]) of the set of belief functions.

In practice, rather than selecting a value  $\alpha$ , one may average out the family of belief functions and build  $\underline{Bel}(C) = \int_0^1 Bel_\alpha(C) d\alpha$ . It comes down to allocating masses  $\underline{m}(F_{\alpha_i}) = b^-(\alpha_i - \alpha_{i+1})$  to  $F_{\alpha_i}$  and  $\underline{m}(\bar{F}_{\alpha_i}) = (1 - b^+)(\alpha_i - \alpha_{i+1})$  to its complement. Note that this approach differs from the previous one ( $Bel \neq \underline{Bel}$ ) because  $\bar{A}_\alpha \neq (\bar{A})_\alpha$ . It is clear that the difference between the two approaches based on belief functions is due to the difference between a fuzzy bipartition in Ruspini sense (a fuzzy set and its complement) used in the fuzzy belief structure, and a fuzzy partition seen as a gradual partition (in the sense of [25]) that to each  $\alpha \in (0, 1]$  associates the crisp partition  $(F_\alpha, \bar{F}_\alpha)$ , a point of view adopted to define a parametric belief function.

## 4.2. Crisp Statements with Fuzzy Probabilities

Let us consider the opposite case, namely  $(A, B)$ , where  $A = [a^-, a^+]$  is a crisp interval and  $B$  is a fuzzy set of probability values (a fuzzy interval on  $[0, 1]$ ). In this situation, Zadeh's definition looks natural: it yields a fuzzy set of probability functions:  $\pi_{(A,B)}(P) = \mu_B(P(A))$ , where  $P(A)$  is the usual probability of  $[a^-, a^+]$ . It is actually a higher-order possibility



distribution over probability functions, as studied in [10]. Alternative approaches may be considered in the spirit of belief functions.

*Fuzzy-valued mass assignment.* One possibility is to see  $([a^-, a^+], B)$  as a belief function with a fuzzy mass function  $\check{m}$  [33]: the mass function  $\check{m}$  associates a fuzzy interval  $\check{m}(E)$  to each focal set  $E$ . Such a fuzzy mass assignment  $\check{m}$  is interpreted as a fuzzy set of belief functions as follows:

$$\mu_{\check{m}}(m) = \min_{E \subseteq U} \mu_{\check{m}(E)}(m(E)).$$

Then the Z-number  $(A, B)$  is viewed as the fuzzy mass function with focal sets  $A, \bar{A}$  defined by  $\check{m}(A) = B$  and  $\check{m}(\bar{A}) = 1 \ominus B$ , using the fuzzy subtraction based on the extension principle. Note that this definition is not consistent with the interpretation of the crisp Z-number  $(A, [b^-, b^+])$ , where  $A$  is crisp. It would assume that the latter defines an interval-valued mass function  $\check{m}(A) = [b^-, b^+]$ ,  $\check{m}(\bar{A}) = [1 - b^+, 1 - b^-]$ , which is too complex to be attractive.

*Z-number as a parametric belief function.* Another option is to consider  $([a^-, a^+], B)$  as a set of crisp Z numbers  $([a^-, a^+], B_\beta)$  parameterised by  $\beta$ . Let  $B_\beta = [b^-(\beta), b^+(\beta)]$ . Each  $([a^-, a^+], B_\beta)$  can be represented by a parametric belief function with mass function  $m_\beta$ , letting  $m_\beta(A) = b^-(\beta)$ ,  $m_\beta(\bar{A}) = 1 - b^+(\beta)$  and  $m_\beta(U) = b^+(\beta) - b^-(\beta)$ , for each choice of  $\beta$ . Averaging out this fuzzy set of belief functions comes down to replacing the fuzzy probability  $B$  by its interval average  $E(B)$  computed as the Aumann integral  $\int_0^1 B_\beta d\beta = [\int_0^1 \inf B_\beta d\beta, \int_0^1 \sup B_\beta d\beta]$  [26, 27]. For instance, consider the discrete case, with membership levels  $\beta_1 > \dots > \beta_\ell > 0$ . The overall weight assigned to  $A$  is then  $\sum_{j=1}^\ell (\beta_j - \beta_{j+1})b^-(\beta) = \inf E(B)$  indeed. This approach is the counterpart to the one in the second part of Section 4.1.

### 4.3. Handling Full-fledged Z-numbers

Suppose now that both  $A$  and  $B$  are fuzzy intervals (i.e. their  $\alpha$ -cuts are closed intervals). It is not easy to propose an interpretation different from the one of Zadeh, that agrees with the preceding cases. We can suggest two approaches.

#### 4.3.1. Hybridising the Previous Cases

One is to put together the fuzzy belief structure of Section 4.1 and the cut approach of Section 4.2. Then we can view both  $A$  and  $B$  as sets of  $\alpha$ -cuts. To each  $A_{\alpha_i}$  is associated a weight  $b^-(\beta)(\alpha_i - \alpha_{i+1})$ ,  $\bar{A}_{\alpha_i}$  is assigned weight  $(1 - b^+(\beta))(\alpha_i - \alpha_{i+1})$ , and  $U$  is assigned weight  $b^+(\beta) - b^-(\beta)$ . We then get a parameterised family of belief functions, with parameter  $\beta$  selecting a probability interval from  $B$ , which may sound hard to use in practice if there is no criterion to select a value  $\beta$ .

However, we can average  $\beta$  out as well. Suppose that  $B$  is a discrete fuzzy set with membership levels  $\beta_1 > \dots > \beta_\ell > 0$  for each  $\beta_j$  the above approach yields a belief function  $Bel_j$ , and we can compute  $Bel = \sum_{j=1}^\ell (\beta_j - \beta_{j+1})Bel_j$  as the representation of the Z-number. It could be proved that this approach comes down to interpreting  $(A, B)$  as a set of  $k\ell$  crisp Z-numbers  $(A_{\alpha_i}, B_{\beta_j})$  each yielding a belief function  $Bel_{ij}$  with mass function

$$m_{ij}(A_{\alpha_i}) = \inf B_{\beta_j}, \quad m_{ij}(\bar{A}_{\alpha_i}) = 1 - \sup B_{\beta_j}, \quad m_{ij}(U) = \sup B_{\beta_j} - \inf B_{\beta_j},$$

and to computing the weighted average

$$Bel = \sum_{i=1}^k \sum_{j=1}^{\ell} (\alpha_i - \alpha_{i+1})(\beta_j - \beta_{j+1}) Bel_{ij}.$$

Again it comes down to interpreting  $(A, B)$  as a crisply qualified fuzzy set  $(A, E(B))$  by the fuzzy belief structure approach of Section 4.1, using the interval average of  $B$ .

#### 4.3.2. The p-box Approach

Another simpler approach is to interpret  $(A, B)$  as a unique belief function representing a generalised p-box. One first idea is to not only view  $A$  as its  $\alpha$ -cuts, but simultaneously use the end-points of the  $\alpha$ -cuts  $[b^-(\alpha), b^+(\alpha)]$  of  $B$  to derive bounds on  $P(A_\alpha)$ . However, since if  $\alpha > \beta$  we get  $P(A_\alpha) \leq P(A_\beta)$ , it is fruitless to interpret  $(A, B)$  as the set of constraints  $b^-(\alpha) \leq P(A_\alpha) \leq b^+(\alpha)$ ,  $0 < \alpha \leq 1$ . Indeed,  $b^-(\alpha)$  increases with  $\alpha$ , while  $P(A_\alpha)$  decreases, so that the lower bounds  $b^-(\alpha)$  are redundant when  $\alpha < 1$ : the set of constraints come down to  $b^-(1) \leq P(A_\alpha) \leq b^+(\alpha)$ ,  $0 < \alpha \leq 1$ .

One way out is to consider  $b^-(1 - \alpha)$  as the lower bound of  $P(A_\alpha)$ , that is we interpret  $B$  as a p-box on  $[0, 1]$  using the pair of decumulative functions

$$\Pi([b, 1]) = \max_{x \geq b} \mu_B(x), \quad N([b, 1]) = \min_{x < b} 1 - \mu_B(x)$$

associated to  $B$ . Using a continuous membership function for  $B$  we have that  $\Pi([b^+(\alpha), 1]) = \mu_B(b^+(\alpha)) = \alpha$  and  $N([b^-(\alpha), 1]) = 1 - \mu_B(b^-(\alpha)) = 1 - \alpha$ .

The set of constraints  $b^-(1 - \alpha) \leq P(A_\alpha) \leq b^+(\alpha)$ ,  $\alpha \in (0, 1]$  forms a generalised p-box on  $U$ , since it is a nested family of subsets whose probabilities are upper and lower bounded [28]. It can be characterised by two possibility distributions  $(\pi^+$  and  $\pi^-)$  built from  $A$  and  $B$  such that  $1 - \pi^- \leq \pi^+$  and  $1 - \pi^-, \pi^+$  are comonotonic functions [29], as we shall detail below. The ordering on  $U$  for generating the cumulative distributions is the one induced by the membership function  $\mu_A$ .

Namely, the set of constraints  $b^-(1 - \alpha) \leq P(A_\alpha)$ ,  $\alpha > 0$  is representable by a possibility distribution  $\pi^+$  on  $U$  such that [16]

$$\pi^+(u) = \min_{u \notin A_\alpha} 1 - b^-(1 - \alpha)$$

In fact, it is easy to see that letting  $A_\alpha = [a^-(\alpha), a^+(\alpha)]$ , we have that  $\pi^+(a^-(\alpha)) = \pi^+(a^+(\alpha)) = 1 - b^-(1 - \alpha)$ . If  $\Pi^+(C) = \sup_{u \in C} \pi^+(u)$  is the possibility measure with distribution  $\pi^+$ , then the convex set of probabilities captured by the set of constraints  $b^-(1 - \alpha) \leq P(A_\alpha)$ ,  $\alpha > 0$  is  $\mathcal{P}(\pi^+) = \{P : P(C) \leq \Pi^+(C), \forall C \text{ measurable}\}$ .

Likewise the set of constraints  $P(A_\alpha) \leq b^+(\alpha)$ ,  $\alpha \in (0, 1]$ , once written as  $P(\overline{A_\alpha}) \geq 1 - b^+(\alpha)$ ,  $\alpha \in (0, 1]$  is representable by a possibility distribution  $\pi^-$  on  $U$  such that

$$\pi^-(u) = \min_{u \in A_\alpha} b^+(\alpha).$$

Again, we have that  $\pi^-(a^-(\alpha)) = \pi^-(a^+(\alpha)) = b^+(\alpha)$ , and the corresponding set of probabilities is  $\mathcal{P}(\pi^-)$ .

We can describe  $\pi^+$  and  $\pi^-$  more precisely:

**Proposition 4.2:** If the support of  $B$  is  $[0, 1]$  and  $\mu_B$  is continuous and concave (in the usual sense) then  $\pi^+(u) \geq \mu_A(u) \geq 1 - \pi^-(u)$  and  $\inf \pi^-(u) = b^+(1)$ .

**Proof:** Indeed suppose  $\mu_A(u) = \alpha$  and, say,  $u = a_\alpha^-$  (the lower bound of the  $\alpha$ -cut of  $A$ ). Then  $\pi^+(u) = \pi^+(a_\alpha^-) = 1 - b^-(1 - \alpha) \geq \alpha$  since from the assumptions on the support and concavity of  $\mu_B$ , function  $1 - \mu_B(\cdot)$  is convex on  $[0, b^-(1)]$ , hence under the line  $1 - \alpha$ , so  $b^-(1 - \alpha) \leq 1 - \alpha$ , hence  $\alpha = \mu_A(a_\alpha^-) \leq \pi^+(a_\alpha^-)$ . Likewise, due to the assumptions on  $\mu_B$ ,  $\pi^-(a_\alpha^-) = b^+(\alpha) \geq 1 - \alpha = 1 - \mu_A(a_\alpha^-)$ . Function  $b^+(\alpha)$  is decreasing with  $\alpha$  and goes from  $b^+(0) = 1$  (since the support of  $B$  is  $[0, 1]$ ) down to  $b^+(1)$ . Hence  $\inf \pi^-(u) = b^+(1)$ , when  $u$  in the core of  $A$ . ■

**Proposition 4.3:** If  $\mu_A(u) = 0$ ,  $\pi^+(u) = 1 - b^-(1)$ ,  $\pi^-(u) = 1$ .

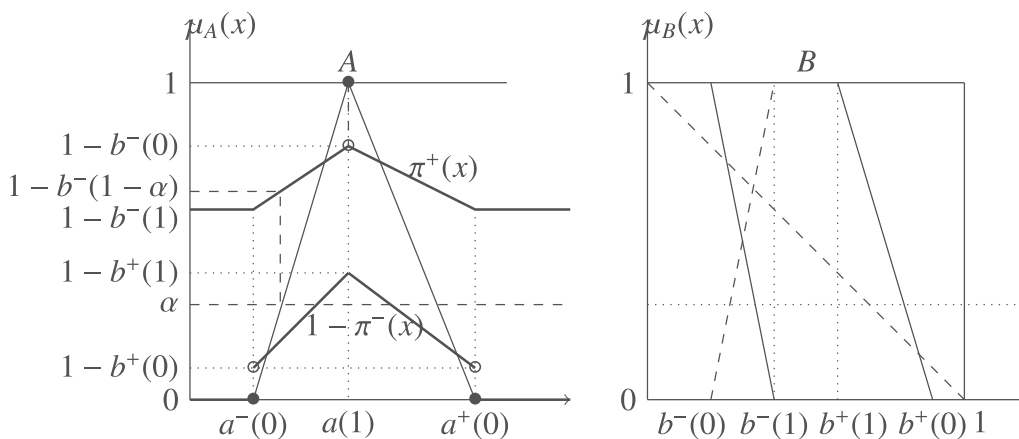
**Proof:** Suppose  $u$  is out of the support of  $A$ . Hence  $u \notin A_\alpha$ ,  $\forall \alpha > 0$ . So  $\pi^-(u) = 1$  as it is the minimum on an empty set. Besides function  $1 - b^-(1 - \alpha)$  is increasing with  $\alpha$  (since  $b^-(\alpha)$  is increasing) and its minimum is attained for  $\alpha = 0$ , which is the case if  $\mu_A(u) = 0$ . ■

Note that the bracketing property  $\pi^+(u) \geq \mu_A(u) \geq 1 - \pi^-(u)$  no longer holds if the support of  $B$  is not  $[0, 1]$ . In particular, if there is a value  $\alpha^* \in (0, 1]$  such that  $1 - b^-(1 - \alpha^*) = \alpha^*$ , then  $\pi^+(a_{\alpha^*}^-) = \mu_A(a_{\alpha^*}^-) = \alpha^*$ . We shall generally have that  $1 - b^-(1 - \alpha) > \alpha$  for  $\alpha > \alpha^*$  so that the  $\alpha$ -cut of  $\pi^+$  is contained in the  $\alpha$ -cut of  $A$  (as shown in Figure 1 in the case when  $A$  and  $B$  are trapezoidal fuzzy numbers).

We can express the two possibility distributions induced by  $(A, B)$  as follows.

$$\pi^+(u) = \begin{cases} 1 & \text{if } \mu_A(u) = 1, \\ 1 - b^-(1 - \mu_A(u)) & \text{if } 0 < \mu_A(u) < 1, \\ 1 - b^-(0) & \text{if } \mu_A(u) = 0. \end{cases}$$

$$\pi^-(u) = \begin{cases} 1 & \text{if } \mu_A(u) = 0, \\ b^+(\mu_A(u)) & \text{if } 0 < \mu_A(u) < 1, \\ b^+(1) & \text{if } \mu_A(u) = 1. \end{cases}$$



**Figure 1.** p-box associated with Z-number  $(A, B)$  (•: point included; ○: point excluded).

Even if  $\pi^+$  and  $1 - \pi^-$  will not always bracket  $\mu_A$ , we do have the inequality  $\pi^+ \geq 1 - \pi^-$  since it comes down to noticing that  $1 - b^-(1 - \alpha) \geq 1 - b^+(1 - \alpha)$ .

The following result can also be established:

**Proposition 4.4:** *The two functions  $\pi^+$  and  $\delta = 1 - \pi^-$  are comonotonic.*

**Proof:** Indeed as both  $A$  and  $B$  are fuzzy intervals, we have that if  $\alpha > \beta$ , then  $[b^-(\alpha), b^+(\alpha)] \subseteq [b^-(\beta), b^+(\beta)]$ . Hence the functions  $b^+(\alpha)$  and  $b^-(1 - \alpha)$  are comonotonic. Now suppose  $\pi^+(u) > \pi^+(v)$  where  $1 > \mu_A(u) > 0$  and  $1 > \mu_A(v) > 0$ . Then  $\pi^+(u) = 1 - b^-(1 - \mu_A(u)) > \pi^+(v) = 1 - b^-(1 - \mu_A(v))$ . Hence  $b^+(\mu_A(u)) \leq b^+(\mu_A(v))$  and we get  $1 - \pi^-(u) = 1 - b^+(\mu_A(u)) \geq 1 - \pi^-(v)$ . ■

As explained in [29], the pair  $(\pi^+, 1 - \pi^-)$  forms a comonotonic cloud [30] corresponding to a set of probabilities  $\mathcal{P} = \mathcal{P}(\pi^+) \cap \mathcal{P}(\pi^-)$ , where  $\mathcal{P}(\pi) = \{P : P(C) \leq \Pi(C), \forall C \text{ measurable}\}$ . This credal set generates a belief function whose focal sets are of the form  $E_\alpha = \{u : \pi^+(u) \geq \alpha\} \setminus \{u : 1 - \pi^-(u) \geq \alpha\}$ . More specifically, in the case of a continuous Z-number, the focal sets obtained are of the following form:

- (1)  $A_1$  with mass  $m(A_1) = b^-(0)$ ;
- (2)  $\{u : \pi^+(u) \geq \alpha\}$  with (infinitesimal) mass  $d\alpha$  for  $1 - b^-(0) \geq \alpha > 1 - b^-(1)$ ;
- (3)  $U$  with mass  $b^+(1) - b^-(1)$ ;
- (4)  $\{u : 1 - \pi^-(u) < \alpha\}$  with (infinitesimal) mass  $d\alpha$  for  $1 - b^+(1) \geq \alpha > 1 - b^+(0)$ ;
- (5)  $\overline{\text{Supp}(A)}$  with mass  $1 - b^+(0)$ .

What we obtain is a (partially) continuous belief function [31].

There are interesting special cases to be noticed.

- In case  $A$  and  $B$  are crisp intervals, the result of the p-box approach degenerates in the belief function of subsection 3. Namely the continuous parts of  $\pi^+$  and  $\pi^-$  disappear since  $1 - b^-(0) = 1 - b^-(1)$  and  $1 - b^+(1) = 1 - b^+(0)$ ;
- If the support of  $B$  is  $[0, 1]$ , some discrete parts of the mass assignment (cases 1 and 5) disappear, and the comonotonic cloud brackets  $\mu_A$ .
- If  $b^+(1) = 1$  ( $B$  expresses a form of certainty) then  $1 - \pi^-(x) = 0$  and only the upper possibility distribution  $\pi^+$  remains. If moreover  $b^-(1) = 1$  then the support of  $\pi^+$  is inside the support of  $A$  (and is equal to it if  $b^-(0) = 0$ ).
- If  $\mu_B(x) = x$  (what could be a genuine gradual representation of probabilistic certainty), it is easy to see that  $\pi^+ = \mu_A$  and  $1 - \pi^-(x) = 0$ . In this case,  $(A, B)$  just reduces to the sure statement  $X$  is  $A$ . This is reminiscent of Zadeh's truth qualification ( $X$  is  $A$  is  $\tau$ ) by the fuzzy truth-value  $\tau$ , he called 'u-true' [3], where  $\mu_\tau(x) = x$  and the result of truth-qualification is of the form  $\mu_\tau(\mu_A)$ . Also in this case, the inequalities  $b^-(1 - \alpha) \leq P(A_\alpha) \leq b^+(\alpha), \alpha \in (0, 1]$  reduce to  $1 - \alpha \leq P(A_\alpha), \alpha \in (0, 1]$ , which is well-known to be a faithful account of the fuzzy number  $A$  (see [32]).
- If  $\mu_B(x) = 1 - x$  (what could be a genuine gradual representation of negative probabilistic certainty, namely that  $X$  is  $A$  is improbable), it is easy to see that  $\pi^+ = 1$  (since  $b^-(0) = b^-(1) = b^+(0) = 0$  and  $b^+(1) = 1$ ) and  $\pi^-(u) = 1 - \mu_A(u)$  since  $b^+(\alpha) = 1 - \alpha$ . It

corresponds to the sure statement  $X$  is  $\bar{A}$ , which is the negation of statement  $X$  is  $A$ . In turn,  $\mu_B(x) = 1 - x$  is reminiscent of Zadeh's truth-qualifier 'u-false' [3].

- If  $A$  is an interval and  $B$  is fuzzy, the inequalities  $b^-(1 - \alpha) \leq P(A_\alpha) \leq b^+(\alpha), \alpha \in (0, 1]$  reduce to the inequalities  $b^-(1) \leq P(A) \leq b^+(1)$ , which is equivalent to the crisp Z-number  $(A, \hat{B})$  using the core of  $B$ .
- if  $B$  is an interval and  $A$  is fuzzy, the inequalities  $b^-(1 - \alpha) \leq P(A_\alpha) \leq b^+(\alpha), \alpha \in (0, 1]$  reduce to the inequalities  $b^- \leq P(A_\alpha) \leq b^+, \alpha \in (0, 1]$ , which is equivalent to the crisp Z-number  $(Supp(A), B)$  using the support of  $A$ . In particular, if  $B = [0, 1]$  (expressing ignorance), it is easy to see that  $\pi^+ = \pi^- = 1$ , which corresponds to complete ignorance about  $A$ .

We notice that we do not retrieve the solutions proposed in Sections 4.1 and 4.2 for cases when only one of  $A, B$  is fuzzy. It suggests that it is not so natural to assign a fuzzy probability to a crisp event or a precise probability to a fuzzy event (in some sense the p-box approach assumes that the gradual nature of  $B$  reflects the gradual nature of  $A$ ).

## 5. Conclusion

The notion of a Z-number is rather naturally found when collecting uncertain information in a linguistic format. It is thus important to propose faithful mathematical representations of this kind of information. In this paper, Z-numbers have been examined in the light of possibility theory, imprecise probabilities and belief functions, in order to provide more solid foundations to this concept. The main message is that it is possible to interpret a Z-number  $(A, B)$  as a special kind of belief function (or random set) on the universe of  $A$  (namely, a p-box), provided that we give up the use of the probability of a fuzzy event, as well as constraints involving the centroid of  $A$ . Indeed the original approach yields a convex (fuzzy) set of probabilities that seems to be very hard to handle in practice. On the contrary, it is much easier to use random sets than convex sets of probabilities induced by any kind of linear constraints. Using the approaches described in this paper, we can compute the uncertainty pervading expressions of the form  $f(X, Y)$  where  $X$  and  $Y$  are Z-numbers by means the random set propagation principle recalled in Section 3 using Monte-Carlo methods (see for instance [18]). Note that the result will not generally be equivalent to another Z-number, but a more general random set, contrary to what some works are presupposing, which does not prevent other Z-numbers on quantities of interest from being extracted from the resulting random set obtained via computation.

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